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A Population Model with Nonlinear Boundary Conditions and Constant Yield Harvesting

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ABSTRACT: We analyze the solutions of a population model with diffusion and strong Allee effect. In particular, we focus our study on a population that satisfies a certain nonlinear boundary condition and on its survival when constant yield harvesting is introduced. We discuss, in detail, results for the one-dimensional case.
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1 INTRODUCTION

Reaction diffusion equations which describe the spatiotemporal distributaries and abundance of organisms are often portrayed by the model

$$u_t = d\triangle u + u\widetilde{f}(x,u)$$

where $u(x,t)$ is the population density, $d > 0$ is the diffusion coefficient, $\triangle u$ is the Laplacian of u with respect to the variable x, and $\widetilde{f}(x,u)$ is the per capita growth rate which is influenced by the heterogenous environment. Skellam first studied such ecology models in [25]. Various reaction diffusion biological models have been studied by [16], previously, though the most classic example is Fisher's equation (see [12]) with $\widetilde{f}(x,u) = (1-u)$. Later, several reaction diffusion models have been used to describe spatiotemporal phenomena in various disciplines such as biology, physics, chemistry, and ecology, (see [5], [11], [21], [22], and [26]). Skellam was the first to use the logistic growth rate $\widetilde{f}(x,u) = m(x) - b(x)u$ in population dynamics to model the crowding effect. However, a general logistic type model can be described by a declining growth rate per capita function, such that $\widetilde{f}(x,u)$ is decreasing with respect to u. The Allee principle, or Allee effect ([1], [2], [10], [19], and [24]), describes an increase in per capita growth rate at low population densities. It can either be strong or weak. For example, if the per capita growth rate is negative at low population densities the Allee effect is strong. On the other hand, if the per capita growth rate is positive at low population densities it is weak. There are many contributing factors to Allee effect in population dynamics, including inbreeding depression, predator saturation, less efficient feeding at low densities, shortage of mates, lack of effective pollination, cooperative behaviors, and reduced effectiveness of vigilance and antipredator defenses.

Several types of nonlinear boundary conditions have been reported in the literature. We examine a boundary in which α, the fraction of individuals who do not cross the boundary when it is reached, is a function of the population density itself. This leads to the following boundary condition

$$\alpha(x,u) = \frac{u}{u - d\nabla u \cdot \eta}$$

or equivalently,

$$d\alpha(x,u)\nabla u \cdot \eta + \left[1 - \alpha(x,u)\right] u = 0 \tag{1}$$

where $\nabla u \cdot \eta$ is the outward normal derivative of u. If $\alpha(x,u) = 0$ then (1) becomes the Dirichlet boundary condition, i.e. all individuals that reach the boundary leave the boundary. While, if $\alpha(x,u) = 1$ then (1) becomes the Neumann boundary condition, i.e. all individuals that reach the boundary remain. This boundary condition has only been recently considered in population dynamics by [5], [6], [7], [14], and [15]. The authors in [17] have also studied a logistic population model with (1).

In this paper, we initiate study of a one-dimensional population model with Strong Allee effect, nonlinear boundary condition, and constant yield harvesting on a bounded domain, $\Omega \subseteq \mathbb{R}$. Throughout the literature, density dependent harvesting has been considered extensively, however, constant yield harvesting is popular in disciplines like fisheries management where harvesting is well regulated. Our main goal is to examine the steady state solutions when $d = 1$ and

$$\alpha(x,u) = \begin{cases} 0; & x = 0 \\ \frac{u}{b}; & x = 1. \end{cases}$$

Namely, we study,

$$\begin{aligned} -u'' &= -u^3 + (a+b)u^2 - abu - c := f(u); \quad (0,1) \\ u(0) &= 0 \\ -u(1)u'(1) &+ \left[u(1) - b\right] u(1) = 0, \end{aligned} \tag{2}$$

where $0 < a < b$, $c \geq 0$ is the constant yield harvesting term, and $\alpha(x, u) : \Omega \text{ x } [0, \infty) \to [0, 1]$ is a smooth, nondecreasing function of u. In the literature, (2) is known as a semipositone problem, since $f(0) < 0$ when $c > 0$. Finding positive solutions to such problems is challenging (see [3], [20], and [23]). Clearly, studying (2) is equivalent to analyzing the two boundary value problems

$$\begin{aligned} -u'' &= -u^3 + (a+b)u^2 - abu - c; \quad x \in (0,1) \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} -u'' &= -u^3 + (a+b)u^2 - abu - c; \quad x \in (0,1) \\ u(0) &= 0 \\ u'(1) &= u(1) - b. \end{aligned} \tag{4}$$

In particular, the positive solutions of (3) and (4) are the positive solutions of (2). Using the Quadrature method of Laetsch (see [18]) the structure of positive solutions of (3) can be found with the aid of Mathematica. In Section 2 we discuss an adaptation of this Quadrature method for (4). Finally, computational results for (2) will be presented in Section 3.

2 QUADRATURE METHOD FOR (4)

The Quadrature method developed by Laetsch (see [18]) has been used extensively in the past by authors such as, [4], [8], [13], and [14], among others. We note that Ladner et. al, adapted the Quadrature method for similar boundary conditions with Logistic growth, see [17]. In this section, we further adapt the Quadrature method to analyze the structure of positive solutions to (4). Define $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$. Suppose that $u(x)$ is a positive solution of (4) with $u'(x_0) = 0$, for some $x_0 \in (0, 1)$. Since (4) is an autonomous differential equation, $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ both satisfy the following initial value problem

$$\begin{aligned} -z'' &= f(z) \\ z(0) &= u(x_0) \\ z'(0) &= 0 \end{aligned} \tag{5}$$

for all $x \in [0, d)$ such that $d = \min\{x_0, 1 - x_0\}$. From this, we have that $u(x_0 + x) \equiv u(x_0 - x)$ as a consequence of Picard's existence and uniqueness theorem. Hence, $u(x)$ is symmetric about x_0 and must have the general shape displayed in Figure 1.

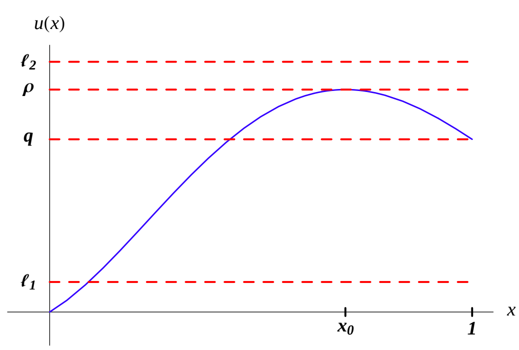

Figure 1: General shape of positive solution to (4).

where $\rho = u(x_0) = \|u\|_\infty$ and $q = u(1)$. It follows that $\ell_1 < \rho < \ell_2$ and $0 \leq q < \rho$ with $u'(x) \geq 0$ for all $x \in [0, x_0]$ and $u'(x) \leq 0$ for all $x \in [x_0, 1]$, where $0 < \ell_1 < \ell_2$ are the positive real zeros of $f(x)$. Multiplying the differential equation in (4) by u' and integrating with respect to x then gives,

$$\frac{-(u')^2}{2} = F(u) + K. \tag{6}$$

Substitution of $x = x_0$ into (6) while using the fact that $u(x_0) = \rho$ and $u'(x_0) = 0$ yields

$$K = F(\rho). \tag{7}$$

Also, substituting $x = 1$ into (6) and recalling that $u(0) = 0$, $u(1) = q$, and $u'(1) = q - b$ gives

$$K = F(q) + \frac{(q-b)^2}{2}. \tag{8}$$

Combining (7) with (8) we have that

$$F(\rho) = F(q) + \frac{(q-b)^2}{2}. \tag{9}$$

Now, solving for u' in (6) yields,

$$u'(x) = \sqrt{2[F(\rho) - F(u)]}; \quad x \in [0, x_0] \tag{10}$$
$$u'(x) = -\sqrt{2[F(\rho) - F(u)]}; \quad x \in [x_0, 1]. \tag{11}$$

This implies that,

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u)}} = \sqrt{2}; \quad x \in [0, x_0) \tag{12}$$
$$\frac{u'(x)}{\sqrt{F(\rho) - F(u)}} = -\sqrt{2}; \quad x \in (x_0, 1]. \tag{13}$$

Integration of (12) and (13) with the use of the first boundary condition of (4) gives

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x; \quad x \in [0, x_0] \tag{14}$$
$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(x - x_0); \quad x \in [x_0, 1]. \tag{15}$$

After substitution of $x = x_0$ in (14) and $x = 1$ in (15), we obtain

$$\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x_0; \quad x \in [0, x_0] \tag{16}$$
$$\int_\rho^{q} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(1 - x_0); \quad x \in [x_0, 1]. \tag{17}$$

Finally, subtraction of (17) from (16) gives,

$$\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2} \tag{18}$$

Notice that in order for $\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$ to be well defined, $F(\rho) > F(s)$ for all $s \in [0, \rho)$. Additionally, the improper integral is convergent if $f(\rho) > 0$. Hence, such a positive solution exists if $f(u)$ and $F(u)$ resemble Figures 2 and 3 respectively,

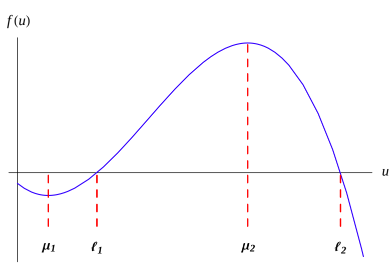

Figure 2: Graph of $f(u)$.

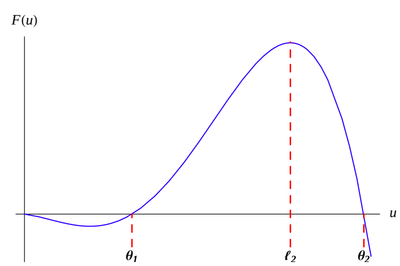

Figure 3: Graph of $F(u)$.

where μ_1, ℓ_i, and θ_i are the zeros of $f'(u)$, $f(u)$, and $F(u)$ respectively for $i = 1, 2$. From these figures, we note that if $\rho \in (\theta_1, \ell_2)$ then both of these conditions hold and the integrals in (18) are well defined. From this and letting (via solving using Mathematica)

$$c_1 := -\frac{1}{27}(a - 2b + \sqrt{a^2 - ab + b^2})(-2a + b + \sqrt{a^2 - ab + b^2})$$
$$(a + b + \sqrt{a^2 - ab + b^2})$$

and

$$c_2 := \frac{\sqrt{2(8a^2 - 11ab + 8b^2)^3} + 2(a + b)(16a^2 - 49ab + 16b^2)}{729}$$

we establish the following:

Theorem 1. *If $b \le 2a$ then (4) has NO positive solution, for any $c \ge 0$.*

Theorem 2. *If $b > 2a$ then (4) has NO positive solution for $c \ge c^*(a, b)$, where $c^*(a, b) = \min\{c_1, c_2\}$.*

Furthermore, we note that given a $\rho > 0$, $x_0 \in (0,1)$ is fixed. Thus we need a unique $q \in [0,\rho)$ corresponding to each ρ-value such that (9) is satisfied. Otherwise, uniqueness of solutions to the initial value problem, (5), would be violated. Let

$$H(x) := F(x) + \frac{(x-b)^2}{2}$$

We see that $H(0) = \frac{b^2}{2}$, $H'(x) = -x^3 + (a+b)^2 + (1-ab)x - c - b$, and $H'(0) = -c - b < 0$. For a unique solution, $q \in [0,\rho)$, of (9) to exist, $H(x)$ must have the structure exemplified in Figure 4,

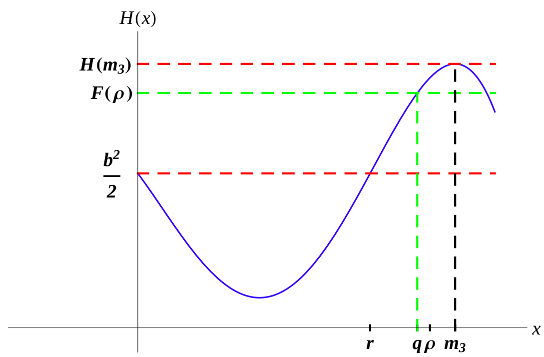

Figure 4: Graph of H(x).

where $m_3(a,b,c)$ is a zero of $H'(x)$. So, for such a unique q to exist $F(\rho) > \frac{b^2}{2}$ must be satisfied. Since $H(\ell_2) > F(\ell_2)$, it follows that this will happen if and only if

$$F(\ell_2) > \frac{b^2}{2}.$$

In particular, $F(\ell_2(a,b,c)) > \frac{b^2}{2}$ must be satisfied. But, $\frac{dF}{dc} = \frac{d\ell_2}{dc} f(\ell_2(a,b,c)) + \int_0^{\ell_2} \frac{\partial f}{\partial c} dt = -\ell_2 < 0$. Hence, it is necessary that $F(\ell_2(a,b,0)) > \frac{b^2}{2}$. After solving using Mathematica, this becomes

$$b > b_1 := a + \sqrt{a^2+6}$$

which leads to the following results:

Theorem 3. *If $b > a + \sqrt{a^2+6}$ then (4) has NO positive solution for $c > c^{**}(a,b)$, where $c^{**}(a,b) < \min\{c_1, c_2\}$ is the unique root of $F(\ell_2(a,b,c)) = \frac{b^2}{2}$.*

Theorem 4. *If $b > a + \sqrt{a^2+6}$ and $c \le c^{**}(a,b)$ then there exists an unique $r(a,b,c) \in (\theta_1, \ell_2)$ such that $F(r) = \frac{b^2}{2}$. Moreover, if $\rho \in [r, \ell_2)$ then*

$$G(\rho, q(\rho)) := \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_{q(\rho)}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

is well defined. In this case, $q = q(\rho) \le \rho$ is the unique solution of $F(\rho) = H(q)$.

We now state and prove this section's main result.

Theorem 5. *Let $b > a + \sqrt{a^2+6}$ and $c \le c^{**}(a,b)$. Then (4) has a positive solution, $u(x)$, with $\|u\|_\infty = \rho$ and $u(1) = q$ if and only if $G(\rho, q) = \sqrt{2}$ for $\rho \in [r, \ell_2)$ and $q \in [0, \rho)$.*

Proof. let $a, b > 0$ s.t. $b > a + \sqrt{a^2+6}$ and $c < c^{**}(a,b)$
($\Rightarrow$:) shown through preceding discussion.
($\Leftarrow$:) suppose: $G(\rho) = \sqrt{2}$ for some $\rho \in [r, \ell_2)$.
Define $u(x) : (0,1) \to \mathbb{R}$ by

$$\begin{aligned} \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} &= \sqrt{2}x; \quad x \in [0, x_0] \\ \int_{\rho}^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} &= -\sqrt{2}(x - x_0); \quad x \in [x_0, 1]. \end{aligned} \tag{19}$$

We now show that $u(x)$ is a positive solution to (4). Clearly, the turning point, x_0, is given by

$$x_0 = \frac{1}{\sqrt{2}} \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}.$$

Note that the function, $\frac{1}{\sqrt{2}}\int_0^{u(x)}\frac{ds}{\sqrt{F(\rho)-F(s)}}$, is a differentiable function of u which is strictly increasing from 0 to x_0 as u increases from 0 to ρ. Thus, for each $x \in [0, x_0]$, there exists a unique $u(x)$ that satisfies

$$\int_0^{u(x)}\frac{ds}{\sqrt{F(\rho)-F(s)}} = \sqrt{2}x. \tag{20}$$

(21)

Moreover, by the Implicit Function theorem, u is differentiable with respect to x. Differentiating (20) gives,

$$u'(x) = \sqrt{2[F(\rho)-F(u)]}; \quad x \in [0, x_0] \tag{22}$$

Similarly, u is a decreasing function of x for $x \in [x_0, 1]$ which yields,

$$u'(x) = -\sqrt{2[F(\rho)-F(u)]}; \quad x \in [x_0, 1] \tag{23}$$

Combining (22) with (23) we arrive at

$$\frac{-(u')^2}{2} = F(\rho) - F(u(x)).$$

Differentiating once more, we have,

$$-u''(x) = f(u(x)).$$

Hence, $u(x)$ satisfies the differential equation in (4). Also, clearly $u(0) = 0$, fulfilling the first boundary condition of (4). Now, from our assumption, $G(\rho) = \sqrt{2}$, it follows that $u(1) = q(\rho)$. Since $F(\rho) = H(q(\rho)) = F(q) + \frac{(q-b)^2}{2}$, we have that

$$\begin{aligned} u'(1) &= -\sqrt{2[F(\rho)-F(q)]} \\ &= -\sqrt{(q-b)^2} \\ &= (q-b) \end{aligned}$$

$$\Rightarrow u'(1) = u(1) - b.$$

Thus, the second boundary condition in (4) is satisfied.
□

3 COMPUTATIONAL RESULTS

In this section, we present computational results to (2) by combining the positive solutions from (3) and (4). For what follows, we are particularly interested in the case when $a = 1$. For (3), the structure of positive solutions is known (see [9] where the authors ascertained the structure of positive solutions via the standard Quadrature method). For (4), we recall Theorem 5 from section 2, and we provide an evolution of the bifurcation curve of positive solutions by plotting the level sets of

$$G(\rho, q) - \sqrt{2} = 0 \tag{24}$$

for $\rho \in [r, \ell_2)$. A numerical root-finding algorithm was implemented in Mathematica to solve equation (24). Due to the nature of the improper integrals in $G(\rho, q)$, the procedure was computationally expensive. Combining results from the two cases, (3) and (4), we are able to analyze the positive solutions of (2). Our computational results for the case $a = 1$ suggest the following results.

Case 1. *For $a = 1$, if $b < b_1$ (some $b_1 \approx 4.77217$) then (2) has NO positive solution for every $c \geq 0$.*

Also, our computations indicate the following existence results for $a = 1$.

Case 2. *For $a = 1$, if $b \in [b_1, b_2]$ (some $b_2 \approx 5.04013$) then there exists a $c_0 > 0$ such that if*

(1) $0 \leq c < c_0$ then (2) has exactly 2 positive solutions.

(2) $c = c_0$ then (2) has a unique positive solution.

(3) $c > c_0$ then (2) has NO positive solution.

Figure 5 shows an example of Case 2.

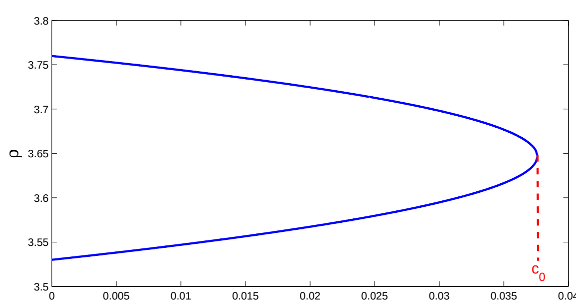

Figure 5: ρ vs c for $a = 1$, $b = 4.78$.

Case 3. *For $a = 1$, if $b \in (b_2, b_3]$ (some $b_3 \approx 5.75907$) then there exist $0 < c_0 < c_1$ such that if*

(1) $c_0 \leq c < c_1$ then (2) has exactly 2 positive solutions.

(2) $0 \leq c < c_0$ or $c = c_1$ then (2) has a unique positive solution.

(3) $c > c_1$ then (2) has NO positive solution.

Figure 6 exemplifies Case 3

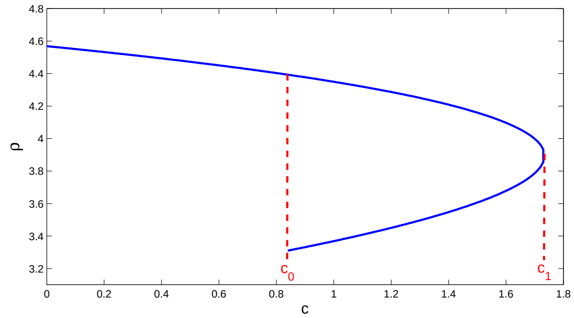

Figure 6: ρ vs c for $a = 1$, $b = 5.1$.

Case 4. *For $a = 1$, if $b \in (b_3, b_4)$ (some $b_4 \approx 7.51988$) then there exists a $c_0 > 0$ such that if*

(1) $0 \leq c \leq c_0$ then (2) has a unique positive solution.

(2) $c > c_0$ then (2) has NO positive solution.

Figure 7 illustrates Case 4.

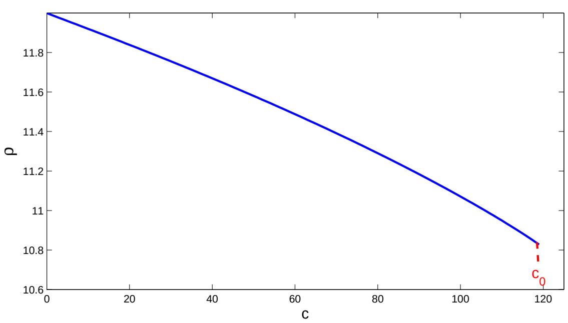

Figure 7: ρ vs c for $a = 1$, $b = 12$.

Case 5. *For $a = 1$, if $b \in [b_4, b_5]$ (some $b_5 \approx 13.128$) then there exist $0 < c_0 < c_1$ such that if*

(1) $0 \leq c < c_0$ then (2) has exactly 3 positive solutions.

(2) $c = c_0$ then (2) has exactly 2 positive solutions.

(3) $c_0 < c \leq c_1$ then (2) has a unique positive solution.

(4) $c > c_1$ then (2) has NO positive solution.

Figures 8 and 9 show examples of this case.

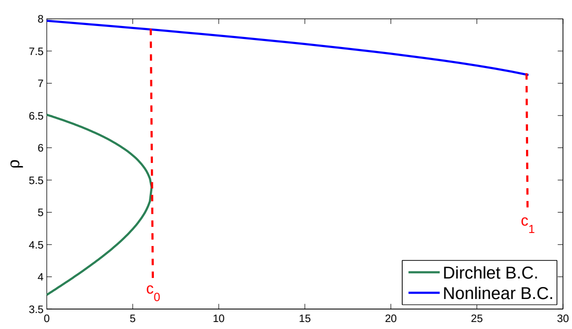

Figure 8: ρ vs c for $a = 1$, $b = 8$.

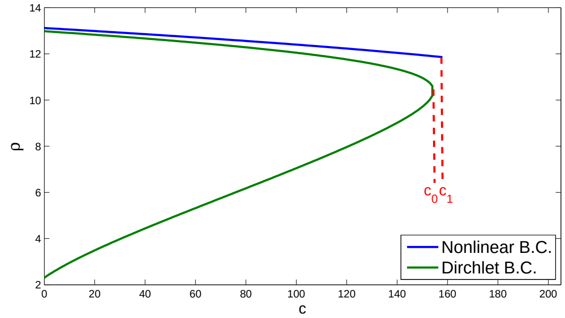

Figure 9: ρ vs c for $a = 1$, $b = 13.12$.

Case 6. *For $a = 1$, if $b \in (b_5, b_6)$ (some $b_6 \in (13.128, 13.128 + \epsilon)$ where $\epsilon > 0$ is small) then there exist $0 < c_0 < c_1 < c_2$ such that if*

(1) $0 \le c \le c_0$ or $c_1 \le c < c_2$ then (2) has exactly 3 positive solutions.

(2) $c_0 < c < c_1$ or $c = c_2$ then (2) has exactly 2 positive solutions.

(3) $c_2 < c \le c_3$ then (2) has a unique positive solution.

(4) $c > c_3$ then (2) has NO positive solution.

Figures 10 and 11 show examples of this case.

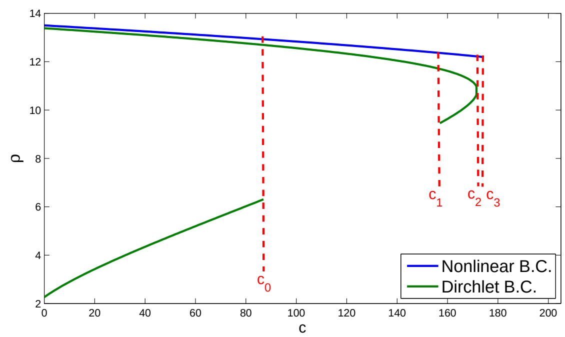

Figure 10: ρ vs c for when $a = 1$, $b = 13.5$.

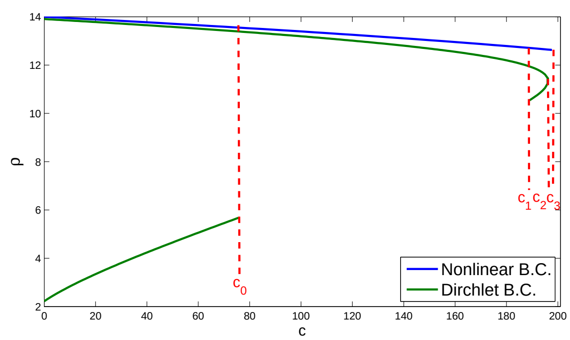

Figure 11: ρ vs c for when $a = 1$, $b = 14$.

Remark 1. *For $b > 14$, we were unable to computationally generate bifurcation curves for (4). This is due to the fact that for b large, the $\rho-$values are too close to their upper bound, ℓ_2.*

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